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COMPUTING GENERAL FORM OF THE FOCAL VALUE AND LYAPUNOV FUNCTION FOR THE LOPSIDED SYSTEM IN DEGREE EIGHT

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Abstract. In this paper, Concerned with planer autonomous system lopsided system of degree eight. We find the general form of all the focal values η_k (k is even and $k \geq 2$) and the Lyapunov function $V(x,y)$ for the lopsided system degree eight. As the type for find the maximum number of limit cycles which can be bifurcate out of the origin and the necessary and sufficient conditions for the existence of center we need to compute the focal values η_{2k+2}, the Lyapunov quantities $L(k)$ and Lyapunov function $V(x,y)$.

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1 Introduction

The computation of Lyapunov quantities is related to its importance in engineering and mechanics of the question on the behaviour of a dynamical system near to the boundary of a stability domain. From (Bautin, 1949) one varies “dangerous” or “safe” limits, i.e. a small alteration of which implies a small (invertible) or noninvertible alterations of the system status correspondingly. Such alterations parallel, for example, to condition of “hard” or “soft” excitations of fluctuations of the system, as shown by Andronov (1966). The development of methods of computation and analysis of Lyapunov quantities (or focus values, Lyapunov coefficients, Poincare-Lyapunov constants) was greatly encouraged by firstly as a pure mathematic problems (such as investigation of stability in critical case of two purely imaginary roots of the first approximation system, Hilbert’s 16th problem, cyclicity problem, and distinguishing between center and focus) and then as to the applied problems (such as the investigation of boundaries of domain of stability and excitation of oscillations). Poincare (1885) and Lyapunov (1966) in their classical works for the analysis of system, conducted the linking of neighbouring boundary of the stability domain and advanced the technique of calculation of the so-called Lyapunov coefficients, (or Lyapunov quantities, focus values, Poincare-Lyapunov constants), which determine the system behavior in the region of the boundary. This method likewise permits us successfully to study the bifurcation of the birth of small cycles (Chavarriga & Grau, 2003; Gine, 2007, Leonov, 2007, 2008), and Yu and Chen (2013). In this work, We find the general form of all the focal values η_k (k is even and $k \geq 2$) and the Lyapunov function $V(x,y)$ for the lopsided system degree eight. As the type for find the maximum number of limit cycles which can be bifurcate out of the origin and the necessary and sufficient conditions for the existence of center we need to compute the focal values η_{2k+2}, the Lyapunov quantities $L(k)$ and Lyapunov function $V(x, y)$.

2 Focal values and Laypunov function for the lopsided system in degree eight

In this section, we introduce the technique of finding the general form of facal values η_{2k} and Layapunov function $V(x, y)$ for as the type for find the maximum number of limit cycles which can be bifurcate out of the origin and the necessary and sufficient conditions for the existence of center, we need to compute the focal values η_{2k+2}, Lyapunov quantities $L(k)$ and Lyapunov function $V(x, y)$. As the way to evaluate the general form for the Lopsided system in degree n. We describe some concepts that we shall need about the related facts and ideas. Which we started by definition of the lopsided system as the following

Definition 1. *(Gine & Santallusia, 2001)*
Suppose that the origin of the system

$$\begin{aligned} \dot{x} &= \lambda x + y \\ \dot{y} &= -x + \lambda y + Q_n(x, y), \end{aligned} \tag{1}$$

where $Q_n(x, y)$ is homogeneous polynomial of degree n and λ is parameter and

$$Q_n(x, y) = \sum_{i=0}^{n} a_i x^{n-i} y^i.$$

Definition 2. *A function $V(x, y)$ is called a Lyapunov function for a system*

$$\begin{aligned} \dot{x} &= \lambda x + y + P(x, y), \\ \dot{y} &= -x + \lambda y + Q(x, y), \end{aligned}$$

where $P(x, y)$ and $Q(x, y)$ are polynomial in the degree n, m respectively. If it satisfies the following conditions:

1. $V(0, 0) = 0$;
2. $V(x, y) > 0$ *in some neighbourhood of the origin;*
3. $\frac{dV}{dt} = \dot{V}$ *is of constant sign in some neighbourhood of the origin.*

Now the function V in a neighbourhood of the origin is such that its rate of change along orbits is of the form

$$\dot{V}(x, y) = \sum_{k=1}^{\infty} \eta_{2k} r^{2k}, \tag{2}$$

where $r^2 = x^2 + y^2$. The coefficients η_{2k} are the focal values and they are polynomials in λ and the coefficients $P(x, y)$ and $Q(x, y)$. It is known that the origin is stable or unstable according to whether the first non-zero focal value is negative or positive, and that the origin is a center if all the focal values are zero.

What we really need are the so-called Lyapunov quantities $L(0), L(1), \ldots, L(K)$, these are the non-zero expressions obtained by calculating each η_{2k} under the condition $\eta_2 = \eta_4 = \ldots = \eta_{2k-2} = 0$. Then the origin is a center if all the Lyapunov quantities are zero. The origin of (1) is said to be a fine focus of order k if $\eta_2 = \eta_4 = \ldots = \eta_{2k} = 0$, but $\eta_{2k+2} \neq 0$. In general $L(k)$ is derived from η_{2k+2}, but it may happen that a reduced focal value is necessarily zero, in which case it does not contribute a Lyapunov quantity.

In this work we choose the lopsided system of degree eight and the parameter λ converse to zero as an example to compute the general form of both facal values η_{2k} and Lyapunov function $V(x, y)$.

The Lyapunov function $V(x, y)$ can be written in the form

$$V_{(x,y)} = \sum_{k=2}^{\infty} v_k$$

and

$$v_k = \sum_{i=0}^{k} v_{k-i,i} x^{k-i} y^i, \tag{3}$$

where $v_2 = \frac{x^2+y^2}{2}$ and v_k is a homogeneous polynomials of degree $k \geq 3$, with unknown coefficients $v_{k-i,i} i + j = k, i, j \geq 0$. For convenience, we say that $v_{k-i,i}$ is an even or odd coefficient according to whether i is even or odd.

Our main result is stated in in the following Theorem where the coefficients $v_{k-i,i} i + j = k, i, j \geq 0$. and the focal values η_k are given.

Theorem 1. *For a lopsided system of degree eight, we have*

1. *The coefficients $v_{k-i,i}$ has the following form*

   (a) *for odd coefficients we have*

$$v_{k-i,i} = \frac{\sum_{j=0}^{\frac{(i-1)}{2}} (k - (2j + 1))!!(2j - 1)!! \left(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k\right)}{(k - i)!!i!!}$$

   (b) *and, for even coefficients we have*

$$v_{k-i,i} = \frac{\sum_{\frac{i}{2}}^{[\frac{k-1}{2}]} (k - (2j + 2))!!j!!\psi_{2j+1}}{(k - i)!!i!!}.$$

2. *The focal values are given by the following formulas*

$$\eta_k = \frac{\sum_{j=0}^{\frac{k}{2}} (k - (2j + 1))!!(2j - 1)!!\psi_{2j}}{\sum_{j=0}^{\frac{k}{2}} (k - (2j + 1))!!(2j - 1)!!\binom{\frac{k}{2}}{j}},$$

   where k (even) and ≥ 4. and

$$\psi_j = \sum_{i=0}^{j+1} i a_{j-i+1} v_{k-7-i,i}.$$

Proof. Note that from the fact that $V(x, y)$ is given as follows,

$$V_{(x,y)} = \sum_{k=2}^{\infty} v_k$$

the derivative of Lyapunov function $V(x, y)$ with respect to the system (1) is given by

$$\begin{aligned} \dot{V} &= \frac{\partial V(x,y)}{\partial x}\dot{x} + \frac{\partial V(x,y)}{\partial y}\dot{y}, \\ &= (x + (v_3)_x + \ldots)(\lambda x + y) + (y + (v_3)_y + \ldots)(-x + \lambda y + Q_8(x, y). \end{aligned}$$

Let D_k be the terms of degree k in $\dot{V}(x,y)$, by direct substitution in the system (1), we get

$$D_k = [y\frac{\partial v_k}{\partial x} - x\frac{\partial v_k}{\partial y}] + [Q_8\frac{\partial v_{k-7}}{\partial y}]. \tag{4}$$

The idea is to choose the coefficients $v_{k-j,j}$ in v_k and the quantities η_k so that

$$D_k = \begin{cases} 0 \text{ if } k \text{ is odd} \\ \eta_k(x^2+y^2)^{k/2} \text{ if } k \text{ is even} \end{cases} \tag{5}$$

When k is odd, the requirement $D_k = 0$ is equivalent to solve a set of $k+1$ unknown and the coefficients arising in the original differential equations.

These $k+1$ equations can be uncoupled into two sets of $\frac{k+1}{2}$ equations, one set determines the odd coefficients of the v_k, and the other determines the even coefficients of v_k.

When k is even, $k = 2m$, the requirement in the second part in (5) gives $k+1$ linear equations for η_k and the $k+1$ coefficients of v_k. These equations also can be uncoupled into two sets: $\frac{k}{2}+1$ equations for η_k and $\frac{k}{2}$ odd coefficients of v_k, and $\frac{k}{2}$ equations for the $\frac{k}{2}+1$ even coefficients of v_k. Then the even coefficients of v_k are uniquely determined under the supplementary conditions $v_{j,j} = 0$ if j is even and $v_{j+1,j-1} + v_{j-1,j+1} = 0$ if j is odd.

Now note that

$$\frac{\partial v_k}{\partial x} = \sum_{i=0}^{k-1}(k-i)v_{k-i,i}x^{k-1-i}y^i, \tag{6}$$

$$\frac{\partial v_k}{\partial y} = \sum_{i=1}^{k} iv_{k-i,i}x^{k-i}y^{i-1}. \tag{7}$$

Therefore

$$\begin{aligned} Q_8\frac{\partial v_{k-7}}{\partial y} &= \sum_{i=0}^{8} a_i x^{8-i}y^i \sum_{i=0}^{k-7} jv_{k-5-j,j}x^{k-5-j-i}y^{i-1} \\ &= \sum_{i=0}^{8}\sum_{j=1}^{k-7} ja_iv_{k-7-j,j}x^{k+1+i-j}y^{i+j-1}. \end{aligned} \tag{8}$$

With the convention that $v_{k-j,j} = 0$ if $j < 0$ or $k-j < 0$.

These last equations in (4) implies

$$D_k = -y\frac{\partial v_k}{\partial x} - x\frac{\partial v_k}{\partial y} + \sum_{i=0}^{8}\sum_{j=1}^{k-7} ja_iv_{k-7-j,j}x^{k+1+i-j}y^{i+j-1}.$$

Now using (6-7) we obtain

$$\begin{aligned} D_k = &\sum_{i=0}^{k-1}(k-i)v_{k-i,i}x^{k-i-1}y^{i+1} - \sum_{i=1}^{k} iv_{k-i,i}x^{k-i+1}y^{i-1} \\ &+ \sum_{i=0}^{8}\sum_{j=1}^{k-7} ja_iv_{k-7-j,j}x^{k+1+i-j}y^{i+j-1}. \end{aligned} \tag{9}$$

We will develop the three terms in D_k each separately

1. The change of index $m = i + 1$ in the first sum in D_k implies

$$\sum_{i=0}^{k-1}(k-i)v_{k-i,i}x^{k-i-1}y^{i+1} = \sum_{m=1}^{k}(k-(m-1))v_{k-(m-1),m-1}x^{k-m}y^{m}.$$

   Since $v_{k-j,j} = 0$ if $j < 0$ or $k - j < 0$, the first sum becomes

$$\sum_{i=0}^{k-1}(k-i)v_{k-i,i}x^{k-i-1}y^{i+1} = \sum_{m=0}^{k}(k-m+1)v_{k-m+1,m-1}x^{k-m}y^{m}. \tag{10}$$

2. If we take $m = i - 1$ in the second sum of D_k, we obtain

$$\sum_{i=1}^{k} iv_{k-i,i}x^{k-i+1}y^{i-1} = \sum_{m=0}^{k-1}(m+1)v_{k-1-m,m+1}x^{k-m}y^{m}. \tag{11}$$

3. In the last sum of D_k we take the index change $m = i + j - 1$, then we obtain

$$\sum_{i=0}^{8}\sum_{j=1}^{k-7} jv_{k-5-i,i}x^{k+i-j+1}y^{i+j-1} = \sum_{j=0}^{k-7}\sum_{m=j-1}^{j+7} ja_{m-j+1}v_{k-7+j,j}x^{k-m}y^{m}.$$

   Developing the sum in the right-hand side and grouping the terms multiplying $x^{k-i}y^i$ for $i = 0, 1, \dots, k$, this sum can be written

$$\sum_{m=j-1}^{j+7}\sum_{j=0}^{k-7} ja_{m-j+1}v_{k-7-j,j}x^{k-m}y^{m} = \sum_{i=0}^{k}\psi_i x^{k-i}y^i, \tag{12}$$

   where

$$\psi_j = \sum_{m=0}^{j+1} ma_{j-m+1}v_{k-7-m,m}. \tag{13}$$

By substituting equations (10),(11) and (12) in equation (9), and we becomes,

$$\begin{aligned} D_k &= \sum_{i=0}^{k}(k-i+1)v_{k-i+1,i-1}x^{k-i}y^i - \sum_{i=0}^{k-1}(i+1)v_{k-i-1,i+1}x^{k-i}y^i + \sum_{i=0}^{k}\psi_i x^{k-i}y^i \\ &= \begin{cases} 0 \text{ if } k \text{ is odd} \\ \sum_{i=0}^{\frac{k}{2}} \binom{\frac{k}{2}}{\frac{i}{2}}\eta_k x^{2(k-i)}y^{2i} \text{ if } k \text{ is even.} \end{cases} \end{aligned} \tag{14}$$

If k is even we have

$$\begin{aligned} D_k &= v_{1,k-1}y^k - v_{k-1,1}x^k + \sum_{i=1}^{k-1}[(k-i+1)v_{k-i+1,i-1} - (i+1)v_{k-i-1,i+1}]x^{k-i}y^i + \sum_{i=0}^{k}\psi_i x^{k-i}y^i \\ &= \sum_{i=0}^{\frac{k}{2}} \binom{\frac{k}{2}}{\frac{i}{2}}\eta_k x^{2(k-i)}y^{2i}. \end{aligned}$$

Thus,

$$\begin{aligned} -v_{k-1,1}+\psi_0 &= \binom{\frac{k}{2}}{0}\eta_k, \\ (k-i+1)v_{k-i+1,i-1}-(i+1)v_{k-i-1,i+1}+\psi_i &= \binom{\frac{k}{2}}{\frac{i}{2}}\eta_k, i=2,4,\cdots,k-2, \text{with } i \text{ is even} \\ (k-i+1)v_{k-i+1,i-1}-(i+1)v_{k-i-1,i+1}+\psi_i &= 0, i=1,3,\cdots,k-1, \text{with } i \text{ is odd} \\ v_{1,k-1}+\psi_k &= \binom{\frac{k}{2}}{\frac{k}{2}}\eta_k. \end{aligned} \tag{15}$$

This implies

$$v_{k-1,1}=-\binom{\frac{k}{2}}{0}\eta_k+\psi_0, \tag{16}$$

and

$$v_{1,k-1}=-\psi_k \tag{17}$$

$$v_{k-i-1,i+1}=\frac{1}{i+1}\left[\psi_i-\binom{\frac{k}{2}}{\frac{i}{2}}\eta_k+(k-i+1)v_{k-i+1,i-1}\right].$$

Then if $m=i-1$

$$v_{k-m,m}=\frac{1}{m}\left[\psi_{m-1}-\binom{\frac{k}{2}}{\frac{m-1}{2}}\eta_k+(k-m+2)v_{k-m+2,m-2}\right], m=1,3,\cdots,k-1, \text{with } m \text{ is od.} \tag{18}$$

and for

$$v_{k-i+1,i-1}=\frac{1}{k-i+1}\left[-\Psi_i+(i+1)v_{k-i-1,i+1}\right].$$

Then if we replace i by $i+1$ we obtain

$$v_{k-i,i}=\frac{1}{k-i}\left[-\psi_{i+1}+(i+2)v_{k-i-2,i+2}\right], i=2,4,\cdots,k-2, \text{with } i \text{ is even.} \tag{19}$$

If k is odd then

$$\begin{aligned} D_k &= \sum_{i=1}^{k}(k-i+1)v_{k-i+1,i-1}x^{k-i}y^i-\sum_{i=0}^{k-1}(i+1)v_{k-i-1,i+1}x^{k-i}y^i+\sum_{i=0}^{k}\psi_i x^{k-i}y^i. \\ &= 0 \end{aligned}$$

Using the similar previous method, then we got the following set of equations

$$v_{k-1,1}=\psi_0, \tag{20}$$

$$v_{1,k-1}=-\psi_k, \tag{21}$$

$$v_{k-i,i}=\frac{-1}{i}[\psi_{i-1}-(k-i+2)v_{k-i+2,i-2}], \tag{22}$$

and

$$v_{k-i,i}=\frac{1}{k-i}[\psi_{i+1}+(i+2)v_{k-i-2,i+2}]. \tag{23}$$

To continue the proof of the theorem we needed to show the following result

Lemma 1. *If k is odd and i is odd from $i = 1, 2, \ldots, k$ then*

$$v_{k-i,i} = \frac{\sum_{j=0}^{\frac{i-1}{2}} (k-(2j+1))!!(2j-1)\psi_{2j}}{(k-i)!!i!!}. \tag{24}$$

Proof. We will show this result using a proof by induction. For $i = 1$ we have

$$v_{k-1,1} = \frac{(k-1)!!(-1)!!\psi_0}{(k-1)!!1!!},$$

Suppose that formula (24) is true for $i = d-2$

$$v_{k-d+2,d-2} = \frac{\sum_{j=0}^{\frac{d-3}{2}} (k-(2j+1))!!(2j-1)!!\psi_{2j}}{(k-d+2)!!(d-2)!!},$$

We must show that it is true for $i = d$, from equation (22) we have

$$\begin{aligned}
v_{k-d,d} &= \frac{-1}{d}\left[\psi_{d-1} - (k-d+2)v_{k-d+2,d-2}\right], \\
&= \frac{-1}{d}\left[\psi_{d-1} - (k-d+2)\frac{\sum_{j=0}^{\frac{d-3}{2}} (k-(2j+1))!!(2j-1)!!\psi_{2j}}{(k-d+2)!!(d-2)!!}\right], \\
&= \frac{-1}{d}\left[\psi_{d-1} - \sum_{j=0}^{\frac{d-3}{2}} \frac{(k-(2j+1))!!(2j-1)!!\psi_{2j}}{(k-d)!!(d-2)!!}\right], \\
&= -\frac{(k-d)!!(d-2)!!\psi_{d-1} - \sum_{j=0}^{\frac{(d-3)}{2}} (k-(2j+1))!!(2j-1)!!\psi_{2j}}{(k-d)!!d!!}, \\
&= \frac{\sum_{j=0}^{\frac{d-1}{2}} (k-(2j+1))!!(2j-1)!!\psi_{2j}}{(k-d)!!d!!}.
\end{aligned}$$

So,

$$v_{k-i,i} = \frac{\sum_{j=0}^{\frac{i-1}{2}} (k-(2j+1))!!(2j-1)!!\psi_{2j}}{(k-i)!!i!!},$$

□

We have also the following result

Lemma 2. *If k is even and i is odd, from $i =, 1, 2, \ldots, k-1$ then*

$$v_{k-i,i} = \frac{\sum_{j=0}^{\frac{i-1}{2}} (k-(2j+1))!!(2j-1)!!(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k)}{(k-i)!!i!!}.$$

Proof. Again, we show this result by induction. For we have $i = 1$

$$\begin{aligned}
v_{k-1,1} &= \frac{(k-1)!!(-1)!!(\psi_0 - \binom{\frac{k}{2}}{0}\eta_k)}{(k-1)!!1!!}. \\
&= -\psi_0 + \binom{\frac{k}{2}}{0}\eta_k.
\end{aligned}$$

Suppose that it's true for $i = d-2$

$$v_{k-d+2,d-2} = \frac{\sum_{j=0}^{\frac{i-3}{2}} (k-(2j+1))!!(2j-1)!!(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k)}{(k-d+2)!!(d-2)!!}.$$

Consider the case $i = d$, from equation (18) we have

$$\begin{aligned}
v_{k-d,d} &= \frac{-1}{d}\left[\psi_{d-1} - \binom{\frac{k}{2}}{\frac{d-1}{2}}\eta_k - (k-d+2)v_{k-d+2,d-2}\right] \\
&= \frac{-1}{d}\left[\psi_{d-1} - \binom{\frac{k}{2}}{\frac{d-1}{2}}\eta_k - (k-d+2)\frac{\sum_{j=0}^{\frac{i-3}{2}}(k-(2j+1))!!(2j-1)!!\left(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k\right)}{(k-d+2)!!(d-2)!!}\right] \\
&= \frac{-1}{d}\left[\psi_{d-1} - \binom{\frac{k}{2}}{\frac{d-1}{2}}\eta_k - \frac{\sum_{j=0}^{\frac{i-3}{2}}(k-(2j+1))!!(2j-1)!!\left(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k\right)}{(k-d)!!(d-2)!!}\right] \\
&= -\frac{(k-d)!!(d-2)!!\left(\psi_{d-1} - \binom{\frac{k}{2}}{\frac{d-1}{2}}\eta_k\right) - \sum_{j=0}^{\frac{i-3}{2}}(k-(2j+1))!!(2j-1)!!\left(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k\right)}{d(k-d)!!(d-2)!!} \\
&= \frac{\sum_{j=0}^{\frac{d-1}{2}}(k-(2j+1))!!(2j-1)!!\left(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k\right)}{(k-d)!!d!!}.
\end{aligned}$$

So,

$$v_{k-i,i} = \frac{\sum_{j=0}^{\frac{i-1}{2}}(k-(2j+1))!!(2j-1)!!\left(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k\right)}{(k-i)!!i!!}.$$

□

We use also a proof by induction to show the following result

Lemma 3. *If k is even or odd and i even from $i = 0, 1, 2, \ldots, k$ then*

$$v_{k-i,i} = \frac{\sum_{j=\frac{i}{2}}^{\lceil\frac{k-1}{2}\rceil}(k-(2j+2))!!2j!!\psi_{2j+1}}{(k-i)!!i!!}.$$

Proof. We have for $i = k$

$$\begin{aligned}
v_{0,k} &= \frac{\sum_{j=\frac{k}{2}}^{\lceil\frac{k-1}{2}\rceil}(k-(2j+2)!!2j!!\psi_{2j+1})}{0!!k!!} \\
&= \frac{(-2)!!k!!\psi_{k+1} + (-1)!!(k-1)!!\psi_k}{k!!} \\
&= A_k,
\end{aligned}$$

where A_k is any constant.

Suppose it's true for $i = d + 2$

$$v_{k-d-2,d+2} = \frac{\sum_{j=\frac{d+2}{2}}^{\lceil\frac{k-1}{2}\rceil}(k-(2j+2)!!2j!!\psi_{2j+1})}{(k-d-2)!!(d+2)!!}.$$

To show that it's true for $i = d$, by using equation19 we obtain,

$$\begin{aligned} v_{k-d,d} &= \frac{1}{k-d}[\psi_{d+1} + (d+2)v_{k-d-2,d+2}] \\ &= \frac{1}{k-d}[\psi_{d+1} + (d+2)\frac{\sum_{j=\frac{d+2}{2}}^{\lceil\frac{(k-1)}{2}\rceil}(k-(2j+2)!!2j!!\psi_{2j+1})}{(k-d-2)!!(d+2)!!}] \\ &= \frac{(k-d)!!d!!\psi_{d+1} + \sum_{j=\frac{d+2}{2}}^{\lceil\frac{k-1}{2}\rceil}(k-(2j+2)!!2j!!\psi_{2j+1})}{(k-d)!!d!!} \\ &= \frac{\sum_{j=\frac{d}{2}}^{\lceil\frac{k-1}{2}\rceil}(k-(2j+2)!!2j!!\psi_{2j+1})}{(k-d)!!d!!}. \end{aligned}$$

So,

$$v_{k-i,i} = \frac{\sum_{j=\frac{i}{2}}^{\lceil\frac{k-1}{2}\rceil}(k-(2j+2)!!2j!!\psi_{2j+1})}{(k-i)!!i!!}.$$

□

Now, return to the proof of Theorem 1. The results of Lemma 1, Lemma 2 and lemma 3 cover cases (a) and (b) in Theorem 1, thus the its first part is proved.

Turn now to the second part of Theorem 1 to prove the formula giving η_k.

From Lamma 2, we have

$$v_{k-i,i} = \frac{\sum_{j=0}^{\frac{i-1}{2}}(k-(2j+1))!!(2j-1)!!(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k)}{(k-i)!!i!!}.$$

After arranging the equation we get,

$$\begin{aligned} (k-i)!!i!!v_{k-i,i} &= \sum_{j=0}^{\frac{i-1}{2}}(k-(2j+1))!!(2j-1)!!\left(\psi_{2j} - \binom{\frac{k}{2}}{j}\eta_k\right) \\ &= \sum_{j=0}^{\frac{i-1}{2}}(k-(2j+1))!!(2j-1)!!\psi_{2j} - \sum_{j=0}^{\frac{i-1}{2}}(k-(2j+1))!!(2j-1)!!\binom{\frac{k}{2}}{j}\eta_k. \end{aligned}$$

Thus we obtain

$$(k-i)!!i!!v_{k-i,i} + \eta_k\sum_{j=0}^{\frac{i-1}{2}}(k-(2j+1))!!(2j-1)!!\binom{\frac{k}{2}}{j} = \sum_{j=0}^{\frac{i-1}{2}}(k-(2j+1))!!(2j-1)!!\psi_{2j}.$$

Taking $i = k+1$, which give $\frac{i-1}{2} = \frac{k}{2}$ in the last equation we obtain

$$\begin{aligned} & 0 + \eta_k\sum_{j=0}^{\frac{k}{2}}(k-(2j+1))!!(2j-1)!!\binom{\frac{k}{2}}{j} \\ &= \sum_{j=0}^{\frac{k}{2}}(k-(2j+1))!!(2j-1)!!\psi_{2j}. \end{aligned}$$

So,

$$\eta_k = \frac{\sum_{j=0}^{\frac{k}{2}}(k-(2j+1))!!(2j-1)!!\psi_{2j}}{\sum_{j=0}^{\frac{k}{2}}(k-(2j+1))!!(2j-1)!!\binom{\frac{k}{2}}{j}}.$$

Which completes the proof of the Theorem 1. □

3 Conclusion

In this paper, a planer autonomous lopsided system of degree eight been studied. Using the classical method of Lyapunov-Poincare, We derived a general form of all the focal values η_k (k is even and $k \geq 2$) and the Lyapunov function $V(x, y)$ for this lopsided system of degree eight. Thus the Lyaponov quantities $L(k)$ can be easily derived and used for the study of the stability of a general dynamic system and calculate the maximum number of limit cycles which can be bifurcate out of the origin for this type of lopsided system.

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